International Journal of Theoretical Physics, Vol. 27, No. 5, 1988

Gauge-Invariant Perturbations in Anisotropic Homogeneous Cosmological Models

Amar Chandra Patra[1] and Dipankar Ray[2]

Received October 23, 1987

In a recent paper K. Tomita and M. Den found a set of coupled differential equations for spatially flat, anisotropic homogeneous, N-dimensional cosmological models. Some particular exact solutions of those differential equations for a few specific equations of state were obtained by D. Lorentz-Petzold. In the present work we solve those differential equations completely.

1. INTRODUCTION

In a recent paper Tomita and Den (1986) gave the following set of coupled differential equations for spatially flat, anisotropic, homogeneous, N-dimensional cosmological models:

$$(N-2)\dot{\alpha}\dot{\beta}_i+\ddot{\beta}_i=0 \tag{1}$$

$$\dot{E}+(E+P)(N-1)\dot{\alpha}=0 \tag{2}$$

$$(N-1)\ddot{\alpha}+\sum_{i=1}^{N-1}(\dot{\beta}_i)^2=-\frac{k}{N-2}[(N-3)E+(N-1)P]\,e^{2\alpha} \tag{3}$$

$$\ddot{\alpha}+(N-2)\dot{\alpha}^2=\frac{k}{N-2}(E-P)\,e^{2\alpha} \tag{4}$$

where the metric of spatially flat, anisotropic, homogeneous models is given by

$$\begin{aligned} ds^2&=g_{\alpha\beta}\,dx^\alpha\,dx^\beta\\ &=e^{2\alpha(\tau)}[-d\tau^2+\gamma_{ij}(\tau)\,dx^i\,dx^j] \end{aligned} \tag{5}$$

[1]Department of Mathematics, Darjeeling Government College, Darjeeling 734101, West Bengal, India.
[2]Department of Mathematics, University of Jadavpur, Calcutta 700032, India.

0020-7748/88/0500-0623$06.00/0

with

$$\gamma_{ii} = \exp[2\beta_i(\tau)], \qquad \sum_{i=1}^{N-1} \beta_i(\tau) = 0$$
$$\gamma_{ij} = 0 \qquad \text{for} \quad i \neq j \tag{6}$$

Here α, β run from 0 to $N-1$, and i, j run from 1 to $N-1$. Here τ is the conformal time and the cosmic time t is related to τ by $dt = e^{\alpha(\tau)}\, d\tau$. Here the universe is assumed to be filled with a perfect fluid and the Einsteins equations

$$G^{\alpha}_{\beta} \equiv R^{\alpha}_{\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta} R = -kT^{\alpha}_{\beta} \tag{7}$$

where the energy-momentum tensor is given by $T^{\mu\nu} = (E+P)u^{\mu}u^{\nu} + pg^{\mu\nu}$ and E and P are the energy density and the pressure, respectively.

Particular exact solutions of the differential equations (1)-(4) for a few specific equations of state were obtained by Lorentz-Petzold (1985) where the β_i have only two different values. In the present note we solve those differential equations completely.

2. SOLUTIONS

Although the above authors gave four differential equations, namely (1)-(4), actually there are three independent equations, which can be seen as follows.

Equation (1) can be integrated to give

$$\dot{\beta}_i = C_i\, e^{-(N-2)\alpha} \tag{8}$$

where C_i are constants of integration.

Equation (2) can be replaced by

$$dE/d\alpha = -(E+P)(N-1) \tag{9}$$

From (3) and (4), one gets

$$(N-1)(N-2)\dot{\alpha}^2 = \sum_{i=1}^{N-1} (\dot{\beta}_i)^2 = 2kE\, e^{2\alpha} \tag{10}$$

It is now easy to verify that both equations (3) and (4) can be obtained by differentiating (10) with respect to α and using (8) and (9). Hence, equations (8)-(10) provide a complete set of differential equations that are equivalent to (1)-(4).

Using equation (8) in equation (10), one obtains

$$(N-1)(N-2)\dot{\alpha}^2 - C\, e^{-2(N-2)\alpha} = 2kE\, e^{2\alpha}, \qquad \text{where } C = \sum_{i=1}^{N-1} C_i^2 \tag{11}$$

Now, to obtain the solutions of equations (8)–(10) one needs a fourth equation namely the equation of state. However, in the present case, we solve the differential equations with an unspecified equation of state as follows.

Let the equation of state be

$$P = P(E) \tag{12}$$

where $P(E)$ is some known function. Equation (9) is solved as

$$\int \frac{dE}{E + P(E)} = -(N-1)\alpha + m_1 \tag{13}$$

where m_1 is a constant of integration.

Equation (11) can now be integrated to give

$$\begin{aligned} m_2 \mp \left(\frac{N-1}{N-2}\right)^{1/2} t &= \int dE \left([E + P(E)] \left\{ C \exp\left[\frac{2(N-2)}{N-1}\left(-m_1 + \int \frac{dE}{E+P(E)}\right)\right] \right.\right. \\ &\quad \left.\left. + 2kE \exp\left[\frac{2}{N-1}\left(m_1 - \int \frac{dE}{E+P(E)}\right)\right]\right\}^{1/2} \right)^{-1} \end{aligned} \tag{14}$$

and equation (8) gives on integration

$$\beta_i = C_i \int \exp\left[-\frac{N-2}{N-1}\left(m_1 - \int \frac{dE}{E+P(E)}\right)\right] dt + D_i \tag{15}$$

where m_2 and D_i are constants of integration.

Thus, the complete set of solutions of the differential equations (1)–(4) is given by (13)–(15). However, the integrals appearing in these equations can only be performed if P is a known function of E.

In the next section, we actually perform these integrals by assuming $P \propto E$.

3. SOLUTION FOR $P \propto E$

Let

$$P = mE, \qquad m = \text{const.} \tag{16}$$

Putting (16) in (13)–(15), one can obtain the solutions of the differential equations (1)–(4) as follows:

$$(N-1)\alpha = m_1 - \frac{\ln E}{m+1} \tag{17}$$

$$C^{1/2}\left[m_2 \mp \left(\frac{N-1}{N-2}\right)^{1/2} t\right] = k_1 \int (z^2-1)^{[1/(m-1)]\{[1/(N-1)]-m\}}\, dz \quad (18)$$

where

$$z^2 = 1 + \frac{2k}{C}\exp\left[(1-m)\left(m_1 - \frac{\ln E}{m+1}\right) + m_1(m+1)\right]$$

$$k_1 = \frac{2}{m-1}\left\{\frac{2k}{C}\exp[m_1(m+1)]\right\}^{(N-2)/(N-1)(m-1)}$$

and

$$\beta_i = \int C_i \exp\left[-\frac{N-2}{N-1}\left(m_1 - \frac{\ln E}{l+m}\right)\right] dt + D_i \quad (19)$$

4. CONCLUSION

In summary, we have shown that of the four differential equations (1)-(4), only three are independent. These three independent equations can be written as equations (8)-(10). The complete set of solutions of these equations is given by (13)-(15) for an unspecified equation of state given by (12).

However, for an equation of state of the form $P \infty E$, i.e., given by (16), one can get solutions of the differential equations (8)-(10) as given by (17)-(19).

REFERENCES

Lorentz-Petzold, D. (1985). *Physical Review D*, **31**, 929.
Tomita, K., and Den, M. (1986). *Physical Review D*, **34**, 3570.